INTRODUCTION TO RATIONAL EXPRESSIONS

EXAMPLE:

You decide to open a small business making gluten-free cakes.
Your start-up costs were $3,000 .
In addition, it costs $10 to produce each cake.
What is the average total cost (ie. including the start-up costs) to produce each cake if you produce x cakes altogether ?

Average total cost = ______________________________

= ______________________________

= ______________

The average total cost above is an example of a rational expression (a polynomial divided by a polynomial).

Rational expressions follow the rules of fractions,
but are more complicated because they don't just involve numbers, but also variables.

Skills which involve rational expressions that we will cover include

- Simplifying
- Multiplying and dividing
- Adding and subtracting (with like or unlike denominators)
- Simplifying complex fractions
- Writing and solving equations

<u>HOW SIMPLIFYING/CANCELLING ACTUALLY WORKS</u>

$$\frac{ac}{bc} = \frac{a}{b}\times\frac{c}{c} = \frac{a}{b}\times 1 = \frac{a}{b}$$

EXAMPLE:

$$\frac{144}{63} = \frac{16\times 9}{7\times 9} = \frac{16}{7}\times\frac{9}{9} = \frac{16}{7}\times 1 = \frac{16}{7} \quad \text{which is often written as} \quad \frac{\cancel{144}\ 16}{\cancel{63}\ 7} = \frac{16}{7}$$

WHY IS THIS INCORRECT ?

$$\frac{3\ \cancel{18}+12}{\cancel{6}} = 3+12 = 15$$

What's happening in the above:

$$\frac{18+12}{6} = \frac{6\times 3+12}{6} = \frac{6\times(3+12)}{6} = \frac{6}{6}\times(3+12) = 1\times(3+12) = 3+12 = 15$$

In reality:

$$\frac{18+12}{6} = \frac{30}{6} = 5 \quad \text{or}$$

$$\frac{18+12}{6} = \frac{6\times 3+6\times 2}{6} = \frac{6\times(3+2)}{6} = \frac{6}{6}\times(3+2) = 1\times(3+2) = 3+2 = 5$$

The result is only correct when you cancel a factor of the numerator against a factor of the denominator. It is not correct to cancel a factor of part of the numerator against a factor of part of the denominator.

The key to simplifying fractions and rational expressions is to factor first.

SIMPLIFYING A RATIONAL EXPRESSION

PROCESS:

1. Factor the numerator and denominator of each expression.
   Do the easier factoring first, so you can get hints on how to do the harder factoring.
   For example, a factor in a numerator may appear as a factor in the denominator.
2. Simplify by cancelling.

The simplification is only valid for values of x for which no denominator was ever 0.

EXAMPLES:

$$\frac{3x^2 - 6x - 24}{x^2 + 5x + 6} =$$

$$\frac{3x^2 + x - 2}{3x^2 - 8x + 4} =$$

MULTIPLYING TWO RATIONAL EXPRESSIONS

PROCESS:

0. Do not multiply the numerators or denominators together yet.

1. Factor the numerator and denominator of each expression.
   Do the easiest factoring first, so you can get hints on how to do the harder factoring.
   For example, a factor in a numerator may appear as a factor in another denominator.
2. Simplify by cancelling.
   You may cancel a factor in any numerator against a factor in any denominator.
3. Multiply together all remaining factors in the numerator and denominator respectively.
   You may leave the numerator and denominator factored in the final answer.

The multiplication is only valid for values of x for which no denominator was ever 0.

EXAMPLES:

$$\frac{x^2-5x-6}{x^2+3x+2}\times\frac{x^2+x-2}{x^2-36} =$$

$$\frac{6x^2-x-12}{2x^2-7x+6}\times\frac{x^2+7x-18}{3x^2-23x-36} =$$

DIVIDING TWO RATIONAL EXPRESSIONS

PROCESS:

1. Replace the divisor with its reciprocal, and replace the division with multiplication.
2. Multiply the dividend by the reciprocal of the divisor.

The division is only valid for values of x for which no denominator was ever 0.

EXAMPLES:

$$\frac{\dfrac{x^2-2x-3}{x^2+x-2}}{\dfrac{x^2-x-2}{x^2-3x+2}} =$$

$$\frac{5x^2-16x+12}{10x^2-17x^2+6x} \div \frac{2x^2-x-6}{14x^2+13x-12} =$$

ADDING / SUBTRACTING TWO RATIONAL EXPRESSIONS WITH LIKE DENOMINATORS

PROCESS:

1. Add / subtract the numerators, and use the common denominator.
   When subtracting, the second numerator must be treated as if it were in a pair of parentheses.
2. Simplify. (The answer might not simplify.)

The addition / subtraction is only valid for values of x for which no denominator was ever 0.

EXAMPLES:

$$\frac{3x^2+9x-5}{3x^2-12}+\frac{17+5x-13x^2}{3x^2-12} =$$

$$\frac{4x^2-5x-15}{x^2+13x-48}-\frac{9-2x-2x^2}{x^2+13x-48} =$$

$$\frac{x^2-8x+2}{x^2-2x-24}+\frac{2x^2-9x-14}{x^2-2x-24} =$$

$$\frac{4x^2-2x+11}{x^2+9x-36}-\frac{7x^2-3x-13}{x^2+9x-36} =$$

FINDING THE LEAST COMMON DENOMINATOR (LCD) OF FRACTIONS

PROCESS:

0. **Do not multiply the denominators together yet.**
1. Factor each denominator completely into prime numbers.
2. Each distinct prime factor in any denominator must appear in the LCD.
3. The power of each prime factor in the LCD is the greatest power of that factor in any denominator.

EXAMPLES:

Find the LCD of $\frac{3}{8}, \frac{7}{54}$

Find the LCD of $\frac{1}{18}, \frac{1}{24}, \frac{3}{40}$

FINDING THE LEAST COMMON DENOMINATOR (LCD) OF RATIONAL EXPRESSIONS

PROCESS:

0. **Do not multiply the denominators together yet.**
1. Factor each denominator completely, including factoring out leading negatives.
2. Each distinct factor in any denominator must appear in the LCD.
   A leading negative can be ignored.
3. The power of each factor in the LCD is the greatest power of that factor in any denominator.

EXAMPLES:

Find the LCD of $\frac{11}{12x^2y^3}, \frac{7}{15x^4y^2}$

Find the LCD of $\frac{6x-5}{24x-36}, \frac{2x}{27-18x}$

ADDING / SUBTRACTING FRACTIONS WITH UNLIKE DENOMINATORS

PROCESS:

1. Find the LCD.
2. For each fraction, determine which factors in the LCD are missing in the original denominator, then multiply the numerator and denominator by the missing factors.
   Carry through the multiplication in the numerator, but keep the denominator in factored form.
3. Add / subtract and simplify.
   The fraction will simplify only if the denominator and the numerator share prime factors.
4. Carry through the multiplication in the denominator for the final answer.

EXAMPLES:

$\frac{3}{16}-\frac{7}{54}=$

$\frac{1}{18}+\frac{1}{24}-\frac{3}{40}=$

ADDING / SUBTRACTING RATIONAL EXPRESSIONS WITH UNLIKE DENOMINATORS

PROCESS:

1. Find the LCD.
2. For each rational expression, determine which factors in the LCD are missing in the original denominator, then multiply the numerator and denominator by the missing factors.
   Carry through the multiplication in the numerator, but keep the denominator in factored form.
3. Add / subtract and simplify.
   The fraction will simplify only if the denominator and the numerator share factors.

The addition / subtraction is only valid for values of x for which no denominator was ever 0.

EXAMPLES:

$$\frac{11}{12x^2y^3} - \frac{7}{15x^4y^2} =$$

$$\frac{6x-5}{24x-36} + \frac{2x}{27-18x} =$$

$$\frac{2x-1}{x^2-x-2} - \frac{x+3}{x^2-4} =$$

$$\frac{2}{7x+5} + \frac{10}{21x+15} - \frac{5}{42x^2+2x-20} =$$

SIMPLIFYING A COMPLEX FRACTION

PROCESS (STANDARD):

1. Add/subtract the numerator and denominator separately.
2. Divide the numerator by the denominator.

The simplification is only valid for values of x for which no denominator was ever 0.

PROCESS (OFTEN FASTER):

1. Find the LCD of all fractions that make up the numerator and denominator.
2. Multiply the numerator and denominator by that LCD.
3. Simplify. (The answer might not simplify.)

The simplification is only valid for values of x for which no denominator was ever 0.

$$\frac{\frac{7}{12}-\frac{1}{8}}{\frac{5}{6}+\frac{1}{4}} =$$

$$\frac{\frac{2}{t-1}+3}{2-\frac{3}{t-1}} =$$

$$\frac{\frac{5}{x+6}-\frac{2}{x}}{\frac{2}{x}-\frac{7}{x+6}} =$$

$$\frac{1-\frac{5}{t-3}}{\frac{1}{t-3}-\frac{3}{t+7}} =$$

SOLVING A PROPORTION

PROCESS:

1. Find the LCD of all fractions & rational expressions in the equation.
2. Multiply both sides of the equation by that LCD.
3. Solve.

$$\frac{15}{4} = \frac{20}{x}$$

$$\frac{x}{9} = \frac{23}{6}$$

$$\frac{21}{20} = \frac{x}{25}$$

$$\frac{32}{x} = \frac{24}{7}$$

SOLVING A RATIONAL EQUATION

PROCESS:

1. Find the LCD of all fractions & rational expressions in the equation.
2. Multiply both sides of the equation by that LCD.
3. Solve.
4. Check answers against original equation to discard extraneous solutions.

$$\frac{10}{x+2}-3 = \frac{10}{x+5}$$

$$\frac{6}{x^2-2x-8}+\frac{5}{x^2+4x+4}=\frac{1}{x-4}$$

$$\frac{1}{3}+\frac{x-3}{2x-10} = \frac{x-4}{x-5}$$

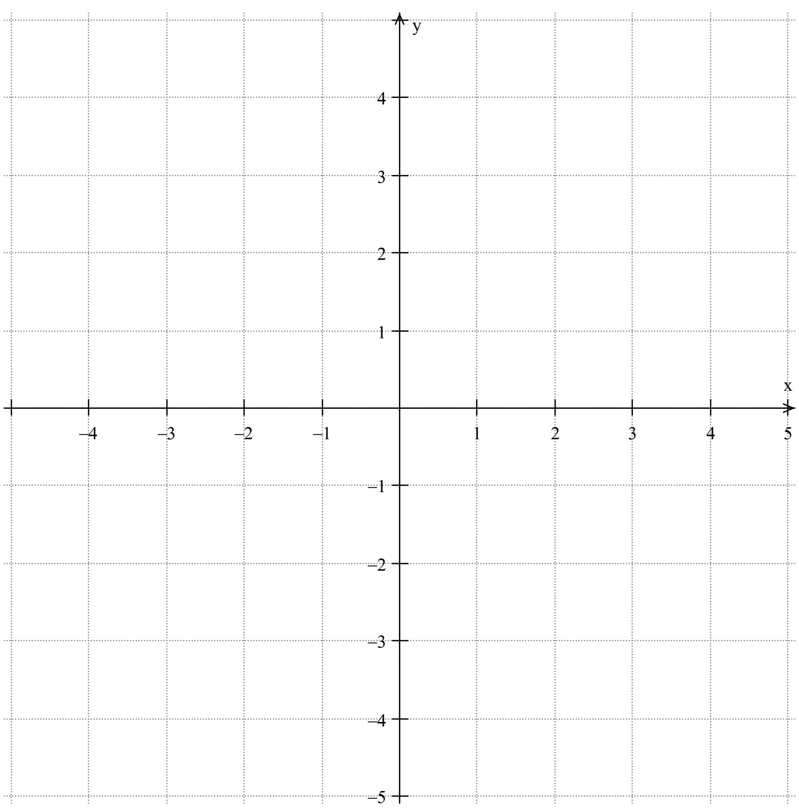
$$\frac{5}{x^2-13x+30}=\frac{7}{x^2-7x-30}-\frac{3}{x^2-9}$$

GRAPHING A RATIONAL FUNCTION AND FINDING ITS ASYMPTOTES

Consider the function $y = \dfrac{2x-5}{x+1}$.

Fill in the table of $y-$values for each of the $x-$values, and plot the corresponding points on the graph paper.

$x =$	$y =$	(x, y)
–5		
–4		
–3		
–2		
–1		
0		
1		
2		
3		
4		
5		



Which $x-$value did not have a corresponding $y-$value ? Why not ?

____

Choose 4 $x-$values (2 greater than, 2 less than) that are very close to the answer of the previous question. Fill in the table of $y-$values for each of those $x-$values, and add the corresponding points to the graph paper.

$x =$	$y =$	(x, y)

The points you plotted show that the graph of the function has a feature you have not seen in the linear and quadratic functions you graphed in earlier algebra classes. That feature is called a **vertical asymptote**.

A vertical asymptote is a vertical line where, as the graph gets closer and closer to the $x-$value of the line, the $y-$values

____

because

____

____

So, in order to find the vertical asymptote, you need to

____

____

Now, look at the $y-$values on the far left and far right sides of the graph.
The $y-$values are getting closer and closer to the value __________
as the $x-$values

____

To see why this happens,
fill in the table of $y-$values (approximately) for each of the following $x-$values.

$x =$	$y =$
100	
1000	
–100	
–1000	

The values you found show that the graph of the function has another feature you have not seen in the linear and quadratic functions you graphed in earlier algebra classes. That feature is called a **horizontal asymptote**.

A horizontal asymptote is a horizontal line where, as the $x-$values of the graph

____

the $y-$values

____

In order to find the horizontal asymptote, you need to

____

____

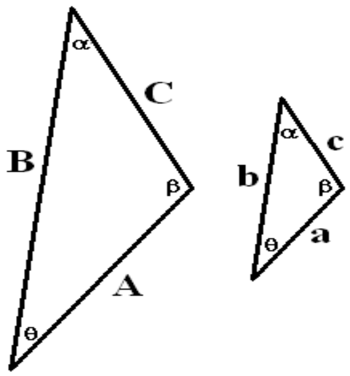
Find the vertical and horizontal asymptotes of the graphs of

$y = \dfrac{4+3x}{5-2x}$

$y = \dfrac{7x+9}{8+6x}$

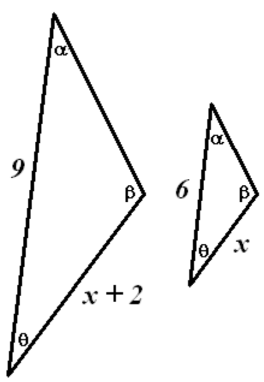
SIMILAR TRIANGLES

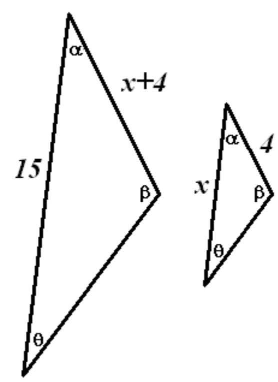
Two triangles are similar if their angles are congruent, and their sides are in proportion.
In fact, if their angles are congruent, two triangles are automatically congruent,
so their sides are automatically in proportion.



$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} \quad \text{OR} \quad \frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

EXAMPLES:





Similar triangles are the basis of trigonometry, and were used in the past to estimate the sizes of difficult to measure objects.

At any particular time of day, all nearby vertical objects and their shadows form similar triangles.
If a 6 foot tall woman casts an 8 foot long shadow, and a nearby building casts a 132 foot shadow,
find the height of the building.

SOLVING AN ABSOLUTE VALUE EQUATION

$|x|$ is the distance of the quantity x from the number 0

$|-6| = 6$

because the quantity -6 is 6 units away from the number 0

or as a diagram,

The equation $|x| = 8$ is equivalent to

the sentence “the quantity x is 8 units away from the number 0” which corresponds to

the diagram which gives

the solutions

The equation $|3x+5| = 9$ is equivalent to

the sentence “the quantity ___________ is _____ units away from the number 0” which corresponds to

the diagram which gives

the solutions

To solve a more complicated equation like $2|3-2x|+7=31$,

first isolate the absolute value,

which is equivalent to

the sentence “the quantity ___________ is _____ units away from the number 0” which corresponds to

the diagram which gives

the solutions

SOLVING AN ABSOLUTE VALUE INEQUALITY

Inequalities involving absolute values
do **NOT** behave like either equations involving absolute values
NOR like inequalities not involving absolute values

INEQUALITIES IN WHICH THE ABSOLUTE VALUE IS LESS THAN A QUANTITY

The inequality $|x| < 5$ is equivalent to

the sentence "the quantity x is less than 5 units away from the number 0" which corresponds to

the diagram which gives

the solution

The inequality $|6 - x| < 7$ is equivalent to

the sentence "the quantity __________ is _______________ units away from the number 0" which corresponds to

the diagram which gives

the solution

To solve a more complicated inequality like $2|x + 4| - 11 < 1$,

first isolate the absolute value,

which is equivalent to

the sentence "the quantity __________ is _______________ units away from the number 0" which corresponds to

the diagram which gives

the solution

INEQUALITIES IN WHICH THE ABSOLUTE VALUE IS GREATER THAN A QUANTITY

The inequality $|x| > 3$ is equivalent to

the sentence "the quantity x is more than 3 units away from the number 0" which corresponds to

the diagram which gives

the solution

The inequality $|4x+1| > 15$ is equivalent to

the sentence "the quantity ___________ is ________________ units away from the number 0" which corresponds to

the diagram which gives

the solution

To solve a more complicated inequality like $5+3|1-x| > 9$,

first isolate the absolute value,

which is equivalent to

the sentence "the quantity ___________ is ________________ units away from the number 0" which corresponds to

the diagram which gives

the solution

Solve $7+2|x-4| > 13$

Solve $11-4|x| > 3$